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Collective radiation and the near-zone field

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Abstract. Collective radiation of several atoms represented by harmonic oscillators is analysed. When taking into account the near-zone field, we show that the existence of the super-radiant mode in the small-system limit is a rare phenomenon. Explicit calculations for four oscillators placed on the vertices of a tetrahedron are presented. Analogous results are obtained for two-level atoms in the case of small total excitation.

1. Introduction

Since the famous paper of Dicke (1954), a lot of work has been done on collective emission from a system of many sources. While the modern version of the theory deals usually with the pencil-shape sample, of dimensions large compared with the wavelength, for which propagation effects play a dominant role (e.g. Bonifacio and Banfi 1975), it is widely believed that in the small-sample limit the original Dicke's description is valid. Namely, in such a limit, only a global dipole moment is coupled to the radiation; the lifetime of the excitation of this degree of freedom is N times shorter than the lifetime of a single atom and the excitations of all the other global modes of the system are trapped and cannot decay through radiative damping. This simple picture can be inadequate, as was suggested in the paper of Friedberg and Hartman (1974a, b) which dealt with a small spherical sample. Recently, this problem was studied for the spherical sample composed of charged harmonic oscillators by Żakowicz (1978). Essential for these studies was taking into account the near-zone field in the system, and longitudinal dipole–dipole forces in particular.

On the other hand, there are a lot of papers studying the problem of two atoms. From the paper of Stephen (1964) it is known that proper collective broadening and narrowing of the emission line conform to a simple picture developed by Dicke.

In our paper we consider the problem of emission from a system of charged harmonic oscillators and show that the situation found for two atoms is quite exceptional and comes from the geometrical simplicity of the system. In § 2 the equations for the problem are derived and general properties of their eigenmodes are deduced. In § 3 the system is further specified to four oscillators located on the vertices of a regular tetrahedron. This simple case can be studied in detail. In § 4 it is shown that the analogous system of two- (or rather four-) level atoms behaves, to a good approximation, like the system of oscillators if the total initial excitation is small.

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2. General remarks

Consider a system of N identical, non-relativistic harmonic oscillators coupled to the electromagnetic field in the dipole approximation. We use the Coulomb gauge, so we have to take into account direct longitudinal dipole–dipole interactions of the oscillators. We assume that centres of harmonic potential are fixed and denoted by $\bar{R}_i$. Our Hamiltonian takes the form

$$H=\sum_{i=1}^{N}\left(\frac{m\bar{\pi}_i^2}{2}+\frac{m\omega_0^2\bar{x}_i^2}{2}\right)+\sum_{i<j}\frac{e^2}{R_{ij}^3}\left[\bar{x}_i\bar{x}_j-3(\bar{n}_{ij}\bar{x}_i)(\bar{n}_{ij}\bar{x}_j)\right]+\sum_{\mu}\int ka_{\bar{k}\mu}^{+}a_{\bar{k}\mu}\,\mathrm{d}_3k$$

where the $\bar{x}_i$ are displacements of charges from their equilibrium positions,

$$R_{ij}=|\bar{R}_i-\bar{R}_j|, \qquad \bar{n}_{ij}=(\bar{R}_i-\bar{R}_j)/|\bar{R}_i-\bar{R}_j|$$

and $a_{\bar{k}\mu}^{\pm}$, $a_{\bar{k}\mu}$ are creation and annihilation operators of photons with definite wavevector $\bar{k}$ and linear polarisation μ. The creation and annihilation operators satisfy the commutation relations

$$[a_{\bar{k}\mu},\, a_{\bar{k}'\mu'}^{+}]=\delta_{\mu\mu'}\delta_{(3)}(\bar{k}-\bar{k}') \qquad [a_{\bar{k}\mu}^{\pm},\, a_{\bar{k}'\mu'}^{\pm}]=0.$$

$\bar{\pi}_i$ are velocity operators for oscillators, and they are related to the canonical momentum and the vector potential by

$$\bar{\pi}_i=(1/m)[\bar{p}_i-e\bar{A}(\bar{R}_i+\bar{x}_i)].$$

We use them instead of canonical variables (or creation and annihilation operators for oscillators) because they possess a better physical sense—they are gauge independent and they are also proper source variables (Rzążewski and Żakowicz 1976) (using them we can show the causality of all the radiative processes). The vector potential has the plane-wave decomposition

$$\bar{A}(\bar{r})=\frac{1}{2\pi}\sum_{\mu}\int \mathrm{d}_3k\,\frac{\bar{\epsilon}_{\bar{k}\mu}}{\sqrt{k}}\,(a_{\bar{k}\mu}\,\mathrm{e}^{\mathrm{i}\bar{k}\bar{r}}+\mathrm{HC}),$$

and in the dipole approximation it is not taken at the actual position of the ith oscillator, but at the position of its centre.

Self-interaction terms, which appear in our model, are ultraviolet divergent in the dipole approximation. We do not hope that one can remove this divergence by a satisfactory renormalisation procedure ('runaways', Norton and Watson 1959), so we introduce a very convenient tempering form-factor, replacing $k^{-1/2}$ in the vector potential, if necessary, by

$$g(k)=[\beta/\sqrt{k}(k^2+\beta^2)^{1/2}],$$

where β is of the order d^{-1}, and d is oscillator size.

The Heisenberg equations for $\bar{x}_i$, $\bar{\pi}_i$, $a_{\bar{k}\mu}^{\pm}$ and $a_{\bar{k}\mu}$ are

$$\begin{aligned}
\dot{a}_{\bar{k}\mu}&=-\mathrm{i}ka_{\bar{k}\mu}+\mathrm{i}\frac{e}{2\pi}\frac{1}{\sqrt{k}}\sum_{j=1}^{N}(\bar{\epsilon}_{\bar{k}\mu}\,\bar{\pi}_j)\exp(-\mathrm{i}\bar{k}\bar{R}_j),\\
a_{\bar{k}\mu}^{\pm}&=\mathrm{i}ka_{k\mu}^{+}-\mathrm{i}\frac{\mathrm{e}}{2\pi}\frac{1}{\sqrt{k}}\sum_{j=1}^{N}(\bar{\epsilon}_{\bar{k}\mu}\bar{\pi}_j)\exp(\mathrm{i}\bar{k}\bar{R}_j),\\
\dot{\bar{x}}_i&=\bar{\pi}_i,
\end{aligned} \tag{1}$$

$$\dot{\bar{\pi}}_i = -\omega_0^2 \bar{x}_i - \frac{e^2}{m} \sum_{j \neq i} \frac{1}{R_{ij}^3} [\bar{x}_j - 3(\bar{x}_j n_{ij}) \bar{n}_{ij}] + \mathrm{i} \frac{e}{2\pi m} \sum_{\mu} \int \mathrm{d}_3 k \sqrt{k} \bar{\epsilon}_{\bar{k}\mu} [a_{\bar{k}\mu} \exp(\mathrm{i}\bar{k}\bar{R}_i) - \mathrm{HC}].$$

We solve this system of equations by performing Laplace transforms of $\bar{a}_{\bar{k}\mu}(t)$, $a^{+}_{\bar{k}\mu}(t)$, $\bar{x}_i(t)$ and $\bar{\pi}_i(t)$, and then after eliminating displacement and field operators we obtain a closed system of linear equations for $\bar{\pi}_i(z)$ (Żakowicz 1978).

$$H(z)\bar{\pi}_i(z) + \sum_{j \neq i} [G(z, R_{ij})\bar{\pi}_j(z) - F(z, R_{ij})(\bar{n}_{ij}\bar{\pi}_j(z))\bar{n}_{ij}]$$

$$= -\omega_0^2 \bar{x}_i(0) + z\bar{\pi}_i(0) - \frac{e^2}{m} \sum_{j \neq i} \frac{1}{R_{ij}^3} [\bar{x}_j(0) - 3(\bar{n}_{ij}\bar{x}_j(0))\bar{n}_{ij}] \tag{2}$$

$$+ \mathrm{i}z \frac{\mathrm{e}}{2\pi m} \sum_{\mu} \int \mathrm{d}_3 k \, k g(k) \bar{\epsilon}_{\bar{k}\mu} \left(\frac{a_{\bar{k}\mu}(0)}{z + \mathrm{i}k} \exp(\mathrm{i}\bar{k}\bar{R}_i) - \mathrm{HC} \right).$$

The RHS of (2) contains only initial data, i.e. initial displacements of oscillators, their velocities and the state of the field. The function $H(z)$ is given by

$$H(z) = z^2 + \omega_0^2 + \frac{4}{3} \frac{e^2}{\pi m} z^2 \int_0^{\infty} \mathrm{d}k \frac{k^3 g^2(k)}{z^2 + k^2},$$

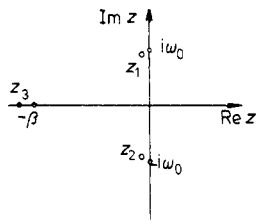
and it is exactly the same as that obtained for the one-oscillator problem (Rzążewski and Żakowicz 1976). $H(z)$ is double valued; one branch is responsible for finding the evolution of one oscillator into the future, while the other one serves for the evolution into the past. Due to our choice of the form-factor the "future" branch has the simple form

$$H(z) = z^2 + \omega_0^2 + \frac{2}{3} \frac{e^2}{m} z^2 \frac{\beta^2}{(z + \beta)},$$

which will be used in further calculations. $H(z)$ has exactly three zeros:

$$z_3 \cong -\beta(1 + 2e^2\beta/3m),$$

$$z_{1,2} \cong \pm \mathrm{i}\omega_0 (1 - e^2\beta/3m) - e^2\omega_0^2/3m.$$



One of them (z_3) plays a role in the solution only for a very short time ($\tau \sim \beta^{-1} \ll \omega_0^{-1}$). Physical processes of damping, emission and scattering are connected with zeros z_1 and z_2. Their real part gives the single-oscillator decay time

$$\tau_0 = |\mathrm{Re}\, z_{1,2}|^{-1} = 3m/e^2\omega_0^2.$$

The functions $G(z, R)$ and $F(z, R)$ are given by

$$G(z, R) = (e^2/mR^3)\, \mathrm{e}^{-zR}(1 + zR + z^2R^2),$$

$$F(z, R) = (e^2/mR^3)\, \mathrm{e}^{-zR}(3 + 3zR + z^2R^2).$$

The factor e^{-zR} ensures retardation effects in the interaction.

To obtain the time-dependent operators we have to perform an inverse Laplace transform, e.g. $\bar{\pi}_i(t) = \int_{\Gamma} (\mathrm{d}z/2\pi\mathrm{i})\, \mathrm{e}^{zt} \bar{\pi}_i(z)$. The contour Γ lies parallel to the imaginary

axis in the complex z plane to the right of all singularities of integrands. We must know then the analytical properties of the z-dependent solutions.

Our closed system of equations for $\bar{\pi}_i(z)$ is a system of $3N$ linear equations:

$$\sum_{\beta=1}^{3N} A_{\alpha\beta}(z)\pi_\beta(z) = K_\alpha(z), \qquad \alpha, \beta = 1, \ldots, 3N,$$

where the matrix $\boldsymbol{A}(z)$ can be written in the form

$$\boldsymbol{A}(z) = H(z)\boldsymbol{I} + (e^2/m)\boldsymbol{B}(z) \qquad (\boldsymbol{I} \text{ is the unit } 3N \times 3N \text{ matrix}).$$

We are especially interested in zeros of the determinant of the matrix $\boldsymbol{A}(z)$, because they can produce the poles of $\pi_\alpha(z)$. Generally, functions of the type $\det \boldsymbol{A}(z)$ can have zeros of three types:

(i) zeros lying far to the left of the imaginary axis, which are important only for a very short time (reconstruction of the transverse field in near wave zone);
(ii) 'modified' zeros of $H(z)$;
(iii) zeros in the right half of the complex plane, which appear in the small-system limit (all $R_{ij} \to 0$).

We expect that only zeros of type (ii) contribute to the relatively slow processes described in this paper (Żakowicz 1978). The 'modified' zeros of $H(z)$, $\tilde{z}_{1,2} = z_{1,2} + \epsilon_{1,2}$, to first order in the small parameter e^2/m, satisfy the following secular equation of degree $3N$:

$$\det\left(2i\omega_0\epsilon_1 \cdot \boldsymbol{I} + \frac{e^2}{m}\boldsymbol{B}(i\omega_0)\right) = 0 \tag{3}$$

(and complex conjugate for ϵ_2).

The following properties of the functions $G(z, R)$ and $F(z, R)$ are important in the small-system limit:

$$\mathrm{Re}\left(\frac{1}{2i\omega_0} G(i\omega_0, R)\right) \xrightarrow[R\to 0]{} \frac{1}{\tau_0} = \frac{e^2\omega_0^2}{3m}, \qquad \mathrm{Re}\left(\frac{1}{2i\omega_0} F(i\omega_0, R)\right) \xrightarrow[R\to 0]{} 0.$$

One can now easily prove that to that order in e^2/m and the small-system limit:

(i) if the homogeneous mode (all $\bar{x}_i(t) = \bar{x}(t)$, $\bar{\pi}_i(t) = \bar{\pi}(t)$ and total dipole moment is not equal to zero) is an eigenmode, then it is a super-radiant mode.

Proof. The z-dependent eigenvalue of the Laplace transform of the coordinate of the homogeneous mode generally takes the form

$$A_{\mathrm{hom,hom}}(z) = H(z) + \sum_{j\neq i} (G(z, R_{ij}) - \alpha_{ij}F(z, R_{ij})),$$

and must be the same for all i.

Zeros $\tilde{z}_{1,2} = z_{1,2} + \epsilon_{1,2}$ of this eigenvalue (which are also the zeros of the whole determinant) are given by

$$2i\omega_0\epsilon_1 + \sum_{j\neq i} (G(i\omega_0, R_{ij}) - \alpha_{ij}F(i\omega_0, R_{ij})) = 0,$$

(and cc for ϵ_2).

So, in the small-system limit,

$$\mathrm{Re}\,\epsilon_{1,2} = -(N-1)\frac{1}{\tau_0}, \qquad \mathrm{Re}\,\tilde{z}_{1,2} = -N \cdot \frac{1}{\tau_0}.$$

Re $z_{1,2}$ is equal to the damping constant of this mode and is N-dependent. It reflects cooperative effects in the emission of radiation. It is obvious that the duration time of the radiation pulse (and also the lifetime for the oscillators) is then N times shorter in comparison with the corresponding time for one oscillator, i.e. if we prepare the system initially in a homogeneous mode we shall obtain a super-radiant emission.

(ii) If the nonhomogeneous mode ($\Sigma_{i=1}^{N} \bar{x}_i(t) = 0$ and total dipole moment equal to zero) is an eigenmode, then it is a non-radiating mode.

Proof. Again we construct the general form of a z-dependent eigenvalue:

$$A_{\text{nhom,nhom}}(z) = H(z) + \sum_{j \neq i} (\beta_{ij} G(z, R_{ij}) - \gamma_{ij} F(z, R_{ij})),$$

and must be the same for all i. Coefficients β_{ij} satisfy $\Sigma_{j \neq i} \beta_{ij} = -1$ in the small-system limit, because $\Sigma_{j \neq i} \bar{x}_j = -\bar{x}_i$ for the nonhomogeneous mode. We obtain

$$\text{Re } \epsilon_{1,2} = 1/\tau_0, \qquad \text{Re } \tilde{z}_{1,2} = 0,$$

so we have radiation trapping in this case.

In general, equation (3) has $3N$ solutions ϵ^{α} for ϵ_1 (and $3N$ complex conjugate $\bar{\epsilon}^{\alpha}$ for ϵ_2). Using the Vieta formulae for the equations of degree $3N$ and the fact that the trace $\text{Tr } \boldsymbol{B}(z) = 0$, one can easily show that

$$\sum_{\alpha=1}^{3N} \epsilon^{\alpha} = 0.$$

This fact leads to the following statement valid in the small-system limit and to first order in e^2/m:

(iii) Solutions of (3) satisfy the following inequalities for all α:

$$-(N-1)/\tau_0 \leqslant \text{Re } \epsilon^{\alpha} \leqslant 1/\tau_0 \qquad \text{or} \qquad -N/\tau_0 \leqslant \text{Re } \tilde{z}^{\alpha} \leqslant 0.$$

Proof. Consider any eigenvector of the matrix $\boldsymbol{B}(\mathrm{i}\omega_0)$. In general it has the form

$$\pi = a(\text{hom}) + b(\text{nhom}),$$

where (hom) and (nhom) are orthonormal homogeneous and nonhomogeneous vectors. If we put $a^2 + b^2 = 1$, then if

$$(\mathrm{e}^2/2\mathrm{i}\omega_0 m)\boldsymbol{B}(\mathrm{i}\omega_0)\pi = -\epsilon\pi$$

then

$$-\epsilon = (e^2/2\mathrm{i}\omega_0 m)\pi \cdot \boldsymbol{B}(\mathrm{i}\omega_0)\pi,$$

where '$\cdot$' denotes scalar product in a $3N$-dimensional vector space. In the small-system limit only the terms coming from the function $G(z, R)$ contribute to the real part of ϵ. We obtain

$$-\text{Re } \epsilon = a^2(N-1)/\tau_0 - b^2/\tau_0,$$

so

$$\text{Re } \epsilon = -Na^2/\tau_0 + 1/\tau_0 \geqslant -(N-1)/\tau_0$$

and

$$\text{Re } \epsilon = -(N-1)/\tau_0 + b^2 N/\tau_0 \leqslant 1/\tau_0.$$

These inequalities reflect the fact that modified zeros of $H(z)$ can lead neither to an antidamping effect nor to a faster than super-radiant effect.

3. Specification of the system

Consider a system consisting of four harmonic oscillators lying on the vertices of a regular tetrahedron. The system of equations for $\bar{\pi}_i(z)$ now takes a simpler form, because of the fact that all R_{ij} are equal ($R_{ij} = R$):

$$H(z)\bar{\pi}_i + \sum_{j\neq i}[(G(z,R) - F(z,R)\bar{n}_{ij}\otimes\bar{n}_{ij})\bar{\pi}_j] = -\omega_0^2\bar{x}_i(0)$$
$$+ z\bar{\pi}_i(0) - \frac{e^2}{mR^3}\sum_{j\neq i}[(\boldsymbol{I} - 3\bar{n}_{ij}\otimes\bar{n}_{ij})\bar{x}_j(0)]$$
$$+ \mathrm{i}z\frac{e}{2\pi m}\sum_{\mu}\int \mathrm{d}_3k\,\sqrt{k}\,\bar{\epsilon}_{\bar{k}\mu}\left(\frac{a_{\bar{k}\mu}(0)}{z+\mathrm{i}k}\exp(\mathrm{i}\bar{k}\bar{R}_i) - \mathrm{HC}\right).$$

Because of the symmetry of our system we are able to diagonalise the matrix $\boldsymbol{A}(z)$. Vectors $(\pi_\alpha(z))$ span the 12-dimensional representation of a symmetry group of a tetrahedron. Solutions belonging to the same k-dimensional invariant subspace transform according to some irreducible representation of the group. We can decompose our 12-dimensional representation into the irreducible representations, decomposing the character of this representation into the characters of irreducible representations (Hamermesh 1962),

$$\Gamma_{\mathrm{osc}} = \Gamma_1 + \Gamma_3 + 2\Gamma_4 + \Gamma_5,$$

where Γ_1 is the character of the one-dimensional identity representation, Γ_3 is the character of the two-dimensional representation, Γ_4 is the character of the three-dimensional vector representation and Γ_5 is the character of the three-dimensional pseudovector representation. Now it is easy to find eigenmodes and their electrostatic eigenfrequencies, given by the solution of the classical problem of four oscillators with dipole–dipole electrostatic interactions. They are as follows (eigenmodes are shown in figure 1):

(a) 'expansion' (nonhomogeneous mode which transforms according to Γ_1):

$$\omega^2 = \omega_0^2 + 5e^2/mR^3 \qquad \omega \cong \omega_0 + \tfrac{5}{2}e^2/mR^3\omega_0$$

(b) 'torsions' (two-dimensional subspace, which transforms according to Γ_3):

$$\omega^2 = \omega_0^2 + \tfrac{1}{2}e^2/mR^3 \qquad \omega \cong \omega_0 + \tfrac{1}{4}e^2/mR^3\omega_0$$

(c) 'rotations' (three-dimensional subspace, which transforms according to Γ_5):

$$\omega^2 = \omega_0^2 - \tfrac{5}{2}e^2/mR^3 \qquad \omega \cong \omega_0 - \tfrac{5}{4}e^2/mR^3\omega_0$$

(d), (e) – we still have a six-dimensional subspace left. In the electrostatic problem this subspace can be decomposed into two three-dimensional subspaces of time-independent eigenvectors, given by linear combinations of two modes: homogeneous (d) and nonhomogeneous (e) (see figure 1).

Thus we have two classical frequencies,

$$\omega^2 = \omega_0^2 + (1+\sqrt{73})e^2/4mR^3, \qquad \omega \cong \omega_0 + 1{\cdot}19e^2/mR^3\omega_0,$$

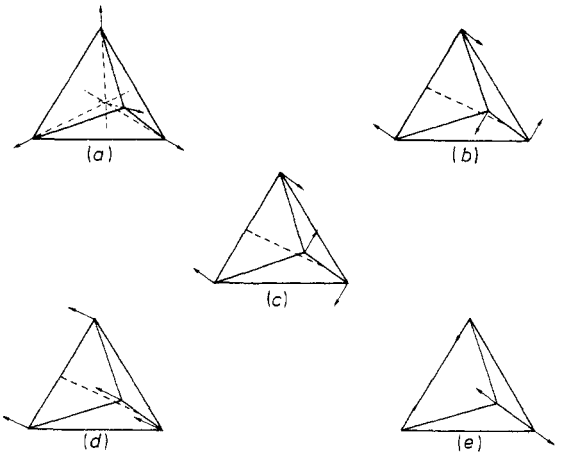


Figure 1. Eigenmodes for the tetrahedron problem: (a) 'expansion', (b) 'torsion', (c) 'rotations', (d) and (e) coupled homogeneous and nonhomogeneous modes.

and

$$\omega^2 = \omega_0^2 + (1-\sqrt{73})e^2/4mR^3, \qquad \omega \cong \omega_0 - 0{\cdot}94e^2/mR^3\omega_0.$$

In a dynamical problem we still have the coupling between these two modes, but the eigenvectors become time-dependent.

It is clear from previous results that the modes (a), (b) and (c) in a small-system limit are non-radiative and have a damping constant equal to zero. Their frequency shift consists of the shift which is an electrostatic collective modification of an eigenmode frequency, and the radiative collective shift. They are given by zeros of the determinant of $\boldsymbol{A}(z)$:

(a) $$\epsilon_1 = \frac{1}{2i\omega_0}(-2F(i\omega_0, R) + G(i\omega_0, R)) \qquad \text{(and CC for } \epsilon_2).$$

In the small-system limit and to first order of e^2/m

$$\tilde{z}_{1,2} \cong \pm i\left(\omega_0 - \frac{e^2\omega_0\beta}{3m} + \frac{5}{2}\frac{e^2}{R^3 m\omega_0} + \frac{3}{4}\frac{e^2\omega_0}{Rm}\right) - O(R^4).$$

(b) $$\epsilon_1 = \frac{1}{2i\omega_0}(G(i\omega_0, R) - \tfrac{1}{2}F(i\omega_0, R)).$$

In the small-system limit,

$$\tilde{z}_{1,2} \cong \pm i\left(\omega_0 - \frac{e^2\omega_0\beta}{3m} + \frac{1}{4}\frac{e^2}{mR^3\omega_0} + \frac{3}{8}\frac{e^2\omega_0}{Rm}\right) - \frac{1}{20}\frac{e^2\omega_0^4 R^2}{m} + O(R^4).$$

(c) $$\epsilon_1 = \frac{1}{2i\omega_0}(G(i\omega_0, R) + \tfrac{1}{2}F(i\omega_0, R)),$$

and, in the small-system limit,

$$\tilde{z}_{1,2} \cong \pm i\left(\omega_0 - \frac{e^2\omega_0\beta}{3m} - \frac{5}{4}\frac{e^2}{mR^3\omega_0} + \frac{1}{8}\frac{e^2\omega_0}{mR}\right) - \frac{1}{48}\frac{\omega_0^4 R^2 e^2}{m} + O(R^4).$$

The coupling between the modes of types (d) and (e) takes the form

$$\begin{pmatrix} H(z)-G(z)+F(z)/2 & -F(z)/\sqrt{2} \\ -F(z)/\sqrt{2} & H(z)+3G(z)-F(z) \end{pmatrix}\begin{pmatrix} v_{\mathrm{nhom}}(z) \\ v_{\mathrm{hom}}(z) \end{pmatrix}$$

$$= \begin{pmatrix} -(\omega_0^2+e^2/2mR^3)q_{\mathrm{nhom}}(0)+(3e^2/\sqrt{2}mR^3)q_{\mathrm{hom}}(0)+zv_{\mathrm{nhom}}(0)+\text{'field part}_{\mathrm{nhom}}\text{'} \\ -\omega_0^2 q_{\mathrm{hom}}(0)+(3e^2/\sqrt{2}mR^3)q_{\mathrm{nhom}}(0)+zv_{\mathrm{hom}}(0)+\text{'field part}_{\mathrm{hom}}\text{'} \end{pmatrix},$$

where $q_{\mathrm{hom}}(q_{\mathrm{nhom}})$, $v_{\mathrm{hom}}(v_{\mathrm{nhom}})$ denote the generalised coordinate and velocity of the homogeneous (nonhomogeneous) mode.

We find four zeros of the determinant of this system to the first order of e^2/m. They are

$$\tilde{z}_{1,2} = z_{1,2}+\epsilon_{1,2}, \qquad \tilde{z}_{3,4} = z_{1,2}+\epsilon_{3,4},$$

where

$$\epsilon_{1,3} = \frac{1}{4i\omega_0}[-2G(i\omega_0, R)+\tfrac{1}{2}F(i\omega_0, R) \pm (16G^2(i\omega_0, R)-12G(i\omega_0, R)F(i\omega_0, R)+\tfrac{17}{4}F^2(i\omega_0, R))^{1/2}]$$

and complex conjugate for $\epsilon_{2,4}$. In the small-system limit we obtain

$$\tilde{z}_{1,2} \cong \pm i\left(\omega_0 - \frac{e^2\omega_0\beta}{3m}+1{\cdot}19\frac{e^2}{mR^3\omega_0}-0{\cdot}06\frac{e^2\omega_0}{mR}\right)-2{\cdot}23\frac{1}{\tau_0},$$

$$\tilde{z}_{3,4} \cong \pm i\left(\omega_0 - \frac{e^2\omega_0\beta}{3m}-0{\cdot}94\frac{e^2}{mR^3\omega_0}-0{\cdot}57\frac{e^2\omega_0}{mR}\right)-1{\cdot}77\frac{1}{\tau_0}.$$

Damping constants are, roughly speaking, two times smaller than they would be in the super-radiant emission.

We have supposed that the system was prepared in such a way that initially only the homogeneous mode was excited, and we have analysed numerically its time-evolution. It is obvious that after some time the coupled nonhomogeneous mode is excited. The energy of the oscillations flows between two modes and is not strictly radiated out of the system. The period of this flow is given by the difference of the imaginary parts of $\tilde{z}_1$ and $\tilde{z}_3$ (in fact it is a simple superposition of the damped oscillations with different frequencies). The period of the flow is much smaller than the lifetime of oscillators for small R, becomes of the order of the lifetime for $R \approx 1/\omega_0$, and then becomes much bigger than the lifetime, so the nonhomogeneous mode has not enough time to gain energy. Damping constants (real parts of $\tilde{z}_1$, $\tilde{z}_3$) become equal to $1/\tau_0$ (the single-oscillator damping constant) for R much bigger than $1/\omega_0$. Figure 2 shows the time-dependence of the radiation intensity.

Similar results can be obtained for a problem of three oscillators lying on the vertices of an equilateral triangle. Figure 3 shows the corresponding eigenmodes. Their frequencies and damping constants are, to the first order in e^2/m, as follows:

(a) homogeneous eigenmode:

$$\epsilon_1 = -(1/2i\omega_0)2G(i\omega_0, R),$$

$$\tilde{z}_{1,2} \cong \pm i\left(\omega_0 - \frac{e^2\omega_0\beta}{3m} + \frac{e^2}{mR^3\omega_0} - \frac{1}{2}\frac{e^2\omega_0}{mR}\right) - 3\frac{1}{\tau_0} + O(R^2).$$

(b) two nonhomogeneous eigenmodes:

$$\epsilon_1 = (1/2i\omega_0)G(i\omega_0, R),$$

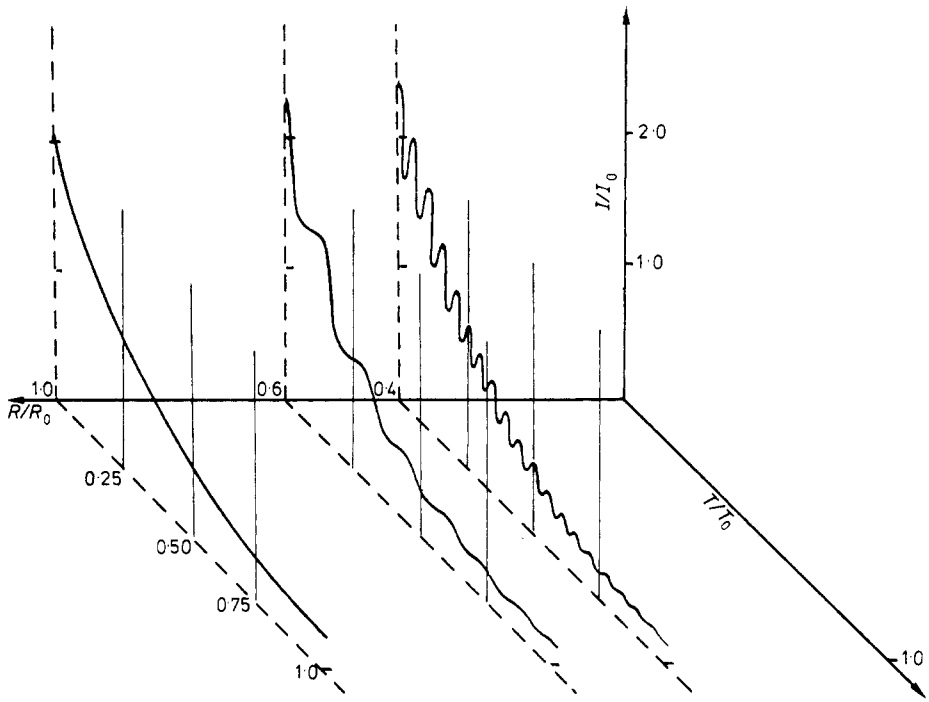


Figure 2. Radiation intensity as a function of time and the distance between oscillators. I_0 – maximal intensity of the single-oscillator radiation; $R_0 = 1/\omega_0 = \lambda_0/2\pi$ where λ_0 is a single-oscillator wavelength; $T_0 = \tau_0/2$.

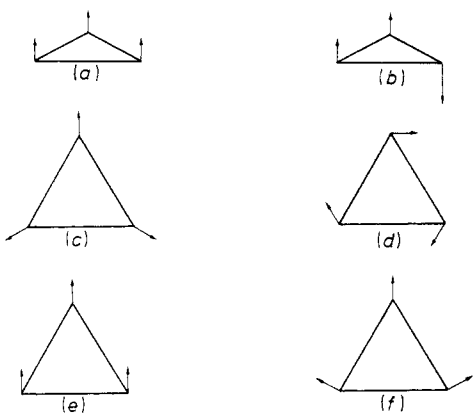


Figure 3. Eigenmodes for the equilatorial triangle problem: (a) homogeneous eigenmode, (b) two nonhomogeneous modes, (c) 'expansion', (d) 'rotation', (e) and (f) coupled homogeneous and nonhomogeneous modes.

and in the small-system limit

$$\tilde{z}_{1,2} \cong \pm i\left(\omega_0 - \frac{e^2\omega_0\beta}{3m} - \frac{1}{2}\frac{e^2}{mR^3\omega_0} + \frac{1}{4}\frac{e^2\omega_0}{mR}\right) - O(R^2).$$

(c) 'expansion':

$$\epsilon_1 = +\frac{1}{2i\omega_0}(G(i\omega_0, R) - \tfrac{3}{2}F(i\omega_0, R)),$$

$$\tilde{z}_{1,2} \cong \pm i\left(\omega_0 - \frac{e^2\omega_0\beta}{3m} + \frac{7}{4}\frac{e^2}{mR^3\omega_0} + \frac{5}{8}\frac{e^2\omega_0}{mR}\right) - O(R^2).$$

(d) 'rotation':

$$\epsilon_{1,2} = (1/2i\omega_0)(G(i\omega_0, R) + \tfrac{1}{2}F(i\omega_0, R)),$$

$$\tilde{z}_{1,2} \cong \pm i\left(\omega_0 - \frac{e^2\omega_0\beta}{3m} - \frac{5}{4}\frac{e}{mR^3\omega_0} + \frac{1}{8}\frac{e^2\omega_0}{mR}\right) - O(R^2).$$

(e) and (f) coupled homogeneous and nonhomogeneous modes:

$$\epsilon_{1,3} = \frac{1}{4i\omega_0}\left[-G(i\omega_0, R) + \frac{F(i\omega_0, R)}{2} \pm (9G^2(i\omega_0, R) - 9G(i\omega_0, R)F(i\omega_0, R) + \tfrac{13}{4}F^2(i\omega_0, R))^{1/2}\right],$$

$$\tilde{z}_{1,2} \cong \pm i\left(\omega_0 - \frac{e^2\omega_0\beta}{3m} - 0{\cdot}96\frac{e^2}{mR^3\omega_0} - 0{\cdot}55\frac{e^2\omega_0}{mR}\right) - 2{\cdot}17\frac{1}{\tau_0},$$

$$\tilde{z}_{3,4} \cong \pm i\left(\omega_0 - \frac{e^2\omega_0\beta}{3m} + 0{\cdot}71\frac{e^2}{mR^3\omega_0} + 0{\cdot}18\frac{e^2\omega_0}{mR}\right) - 0{\cdot}83\frac{1}{\tau_0}.$$

4. Equations of the four-level atoms system

In this section we shall discuss briefly the case of more realistic atoms. To obtain a nice rotational structure for an atom we shall consider four-level atoms rather than two-level ones. We assume that their ground level is an s state (denoted by $|-\rangle$) while the excited level is a p state degenerated with respect to the magnetic quantum number $m = \pm 1, 0$. Instead of using eigenstates of L_z, we use the states $|+\rangle_x$, $|+\rangle_y$ and $|+\rangle_z$ for which a matrix element of a dipole moment $\langle + |\vec{d}| - \rangle$ of an atom is parallel to the directions of x, y, z respectively.

We describe our system by the Hamiltonian in the dipole approximation with the $\tilde{p}\,.\,\tilde{A}$ coupling, and we take into account dipole–dipole interactions between atoms (longitudinal part of the electric field).

The Hamiltonian takes the form

$$H = \sum_{i=1}^{N}\frac{\omega_0}{2}\sigma^3(i) + \sum_{\mu}\int d_3k\, k a^{+}_{\bar{k}\mu}a_{\bar{k}\mu}$$
$$+ \sum_{i<j}\frac{d^2}{R_{ij}^3}\{(\bar{\sigma}^{+}(i) + \bar{\sigma}(i))(\bar{\sigma}^{+}(j) + \bar{\sigma}(j)) - 3[(\bar{\sigma}^{+}(i) + \bar{\sigma}(i))\,.\,\bar{n}_{ij}][(\bar{\sigma}^{+}j) + \bar{\sigma}(j))\,.\,\bar{n}_{ij}]\}$$
$$+ \sum_{i=1}^{N}\frac{e^2}{2m}\bar{A}(\bar{R}_i) - i\sum_{i=1}^{N}\sum_{\mu}\int d_3k\frac{\omega_0 d}{2\pi\sqrt{k}}\bar{\epsilon}_{\bar{k}\mu}(\bar{\sigma}^{+}(i) - \bar{\sigma}(i))[a_{\bar{k}\mu}\exp(i\bar{k}\bar{R}_i) - \text{HC}],$$

where i, j denote the ith, jth atom,

$$\sigma^3 = \sum_\alpha |+\rangle_{\alpha\alpha}\langle +|-|-\rangle\langle -|$$

is an operator of the energy of a free atom,

$$\bar{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$$

and

$$\sigma^\alpha = |-\rangle_\alpha\langle +|$$

is an operator which changes the upper state $|+\rangle_\alpha$ into the ground state. d denotes the value of the dipole moment of the transition considered.

Inspired by the Wigner–Weisskopf (Weisskopf and Wigner 1930) approximation, extending the considerations of Milloni and Knight (1974), we restrict ourselves to the subspace of states spanned by the following orthogonal vectors:

$$|i, \alpha\rangle = |-\rangle\ldots\underset{i\text{th atom}}{|-\rangle|+\rangle_\alpha}|-\rangle\cdots|-\rangle|\Omega_{\text{ph}}\rangle,$$

$$\text{energy } E = -\tfrac{1}{2}(N-2)\omega_0,$$

$$|\bar{k}, \mu\rangle = |-\rangle\ldots|-\rangle|1\bar{k}\mu\rangle, \qquad \text{energy } E = k - N\omega_0/2,$$

$$|1\bar{k}\mu, i\alpha, j\beta\rangle = |-\rangle\ldots\underset{i\text{th atom}}{|-\rangle|+\rangle_\alpha}|-\rangle\ldots\underset{j\text{th atom}}{|-\rangle|+\rangle_\beta}|-\rangle\ldots|-\rangle|1\bar{k}\mu\rangle,$$

$$E = k - (N-4)\omega_0/2.$$

In a semiclassical Bloch picture it corresponds to the small oscillations of the Bloch vector around its equilibrium position. Using this subspace, the $\bar{A}^2$ term produces merely a small shift of the photon frequency, so we shall neglect it.

We analyse the time evolution of the state:

$$|\psi(t)\rangle = \sum_{i,\alpha} b_i^\alpha(t)|i, \alpha\rangle + \sum_\mu \int \mathrm{d}_3 k\, b(\bar{k}, \mu, t)|\bar{k}, \mu\rangle$$

$$+ \sum_{\substack{i,\alpha,j,\beta \\ i<j}} \sum_\mu \int \mathrm{d}_3 k\, c_{ij}^{\alpha\beta}(\bar{k}, \mu, t)|i, \alpha, j, \beta, 1\bar{k}, \mu\rangle,$$

using the Schrödinger equation and orthogonality properties. Equations for the time-dependent coefficients take the form

$$\dot{b}_i^\alpha(t) = \mathrm{i}\left(\frac{N-2}{2}\right)\omega_0 b_i^\alpha(t) - \mathrm{i}\sum_{j\neq i}\frac{d^2}{R_{ij}^3}[b_j^\alpha - 3n_{ij}^\alpha(\bar{b}_j(t)\bar{n}_{ij})]$$

$$- \frac{d\omega_0}{2\pi}\sum_\mu \int \mathrm{d}_3 k \frac{1}{\sqrt{k}}\exp(\mathrm{i}\bar{k}\bar{R}_i)\epsilon_{k_\mu}^\alpha b(\bar{k}, \mu, t)$$

$$+ \frac{d\omega_0}{2\pi}\sum_{j\neq i}\sum_\mu \int \mathrm{d}_3 k \frac{1}{\sqrt{k}}\exp(\mathrm{i}\bar{k}\bar{R}_j)\left(\sum_\beta \epsilon_{k\mu}^\beta c_{ij}^{\alpha\beta}(\bar{k}, \mu, t)\right), \tag{4}$$

$$\dot{b}(\bar{k}, \mu, t) = -\mathrm{i}\left(k - \frac{N}{2}\omega_0\right) + \frac{d\omega_0}{2\pi}\sum_{j=1}^N \frac{1}{\sqrt{k}}(\bar{\epsilon}_{k\mu}\bar{b}_j(t))\exp(-\mathrm{i}\bar{k}\bar{R}_j)$$

$$- \mathrm{i}\sum_{j>i}\frac{d^2}{R_{ij}^3}\left\{\sum_\beta\left[c_{ij}^{\beta\beta}(t) - 3\sum_\alpha (c_{ij}^{\alpha\beta}(t)n_{ij}^\alpha n_{ij}^\beta)\right]\right\},$$

$$\dot{c}_{ij}(\bar{k},\mu,t)=-\mathrm{i}(k-(N-4)\,\omega_0/2)c_{ij}^{\alpha\beta}(\bar{k},\mu,t)-(d\omega_0/2\pi\sqrt{k})\epsilon_{k\mu}^{\beta}b_i^{\alpha}(t)\exp(-\mathrm{i}\bar{k}\bar{R}_j)$$
$$-\frac{d\omega_0}{2\pi}\frac{1}{\sqrt{k}}\epsilon_{k\mu}^{\alpha}b_j^{\beta}(t)\exp(-\mathrm{i}\bar{k}\bar{R}_i)-\mathrm{i}\sum_{\substack{k\neq i\\k\neq j}}\frac{d^2}{R_{jk}^3}\left(c_{ik}^{\alpha\beta}-3\sum_{\gamma}c_{ik}^{\alpha\gamma}n_{jk}^{\gamma}n_{jk}^{\beta}\right)$$
$$-\mathrm{i}\sum_{\substack{k\neq i\\k\neq j}}\frac{d^2}{R_{ik}^3}\left(c_{kj}^{\alpha\beta}-3\sum_{\gamma}c_{kj}^{\gamma\beta}n_{ki}^{\gamma}n_{ki}^{\alpha}\right)-\mathrm{i}\frac{d^2}{R_{ij}^3}(\delta^{\alpha\beta}-3n_{ij}^{\alpha}n_{ij}^{\beta})b(t).$$

We then perform Laplace transforms and eliminate $b(\bar{k},\mu,z)$ and $c_{ij}^{\alpha\beta}(\bar{k},\mu,z)$, restricting ourselves to the terms of second order in the coupling constant $d\omega_0$. We do not have to solve difficult equations for $b(\bar{k},\mu,z)$ and $c_{ij}^{\alpha\beta}(\bar{k},\mu,z)$ exactly—in fact we need only first-order solutions (i.e. technically we neglect 'mixed' terms of order $d^2\omega_0^2$ in the equations for $b(\bar{k},\mu,z)$ and $c_{ij}^{\alpha\beta}(\bar{k},\mu,z)$). Finally we obtain a closed system of linear equations for $b_i^{\alpha}(z)$, which can be written in the form

$$\bar{b}_i(z)=(b_i^x(z),b_i^y(z),b_i^z(z)),$$
$$h(z)\bar{b}_i(z)+\sum_{j\neq i}[(g(z,R_{ij})-f(z,R_{ij})\bar{n}_{ij}\otimes\bar{n}_{ij})\bar{b}_j(z)]$$
$$=\bar{b}_i(0)-\frac{d\omega_0}{2\pi}\sum_{\mu}\int \mathrm{d}_3k\frac{1}{\sqrt{k}}\exp(\mathrm{i}\bar{k}\bar{R}_i)\bar{\epsilon}_{\bar{k}\mu}\cdot\frac{b(\bar{k},\mu,0)}{z+\mathrm{i}(k-N\omega_0/2)}$$
$$+\frac{d\omega_0}{2\pi}\sum_{j\neq i}\sum_{\mu}\int \mathrm{d}_3k\frac{1}{\sqrt{k}}\exp(\mathrm{i}\bar{k}\bar{R}_j)\sum_{\beta}\epsilon_{k\mu}^{\beta}\frac{c_{ij}^{\alpha\beta}(\bar{k},\mu,0)}{z+\mathrm{i}(k-(N-4)\omega_0/2)}.$$

The function $h(z)$ is similar to the one obtained in the one-atom problem, and it is given by

$$h(z)=z-\mathrm{i}\frac{N-2}{2}\omega_0+\frac{2d^2\omega_0^2}{3\pi}\int_0^{\infty}\frac{k^2g(k)^2\,\mathrm{d}k}{z+\mathrm{i}(k-N\omega_0/2)}$$
$$+\frac{2}{3}d^2\frac{\omega_0^2}{\pi}(3N-3)\int_0^{\infty}\frac{k^2g(k)^2\,\mathrm{d}k}{z+\mathrm{i}(k-(N-4)\omega_0/2)}$$

where we introduced the form-factor $g(k)$ because of the divergence of the self-interaction terms. The functions $g(z,R)$, and $f(z,R)$ have the form

$$g(z,R)=\mathrm{i}\frac{d^2}{R^3}+\frac{d^2\omega_0^2}{\pi}\int_0^{\infty}\frac{k\,\mathrm{d}k\tilde{g}(k,R)}{z+\mathrm{i}(k-N\omega_0/2)}+\frac{d^2\omega_0^2}{\pi}\int_0^{\infty}\frac{k\,\mathrm{d}k\,\tilde{g}(k,R)}{z+\mathrm{i}(k-(N-4)\omega_0/2)}$$

and

$$\tilde{g}(k,R)=\frac{\sin kR}{kR}+\frac{\cos kR}{k^2R^2}-\frac{\sin kR}{k^3R^3},$$
$$f(z,R)=3\mathrm{i}\frac{d^2}{R^3}+\frac{d^2\omega_0^2}{\pi}\int_0^{\infty}k\,\mathrm{d}k\tilde{f}(k,R)\left(\frac{1}{z+\mathrm{i}(k-N\omega_0/2)}+\frac{1}{z+\mathrm{i}(k-(N-4)\omega_0/2)}\right)$$

and

$$\tilde{f}(k,R)=\frac{\sin kR}{kR}+\frac{3\cos kR}{k^2R^2}-\frac{3\sin kR}{k^3R^3}.$$

Now we make the so-called pole approximation i.e. we neglect the z-dependence in the

integrals appearing in $h(z)$, $g(z, R)$ and $f(z, R)$, putting

$$z = \mathrm{i}(N-2)\omega_0/2 + \epsilon, \qquad (\text{small } \epsilon > 0).$$

We can then perform the integrals in $g(z, R)$ and $f(z, R)$ exactly. The term coming from static dipole–dipole interactions of atoms is cancelled by the pole $1/k$ in the integrals. To obtain $h(z)$ we use the formula $1/(x \pm \mathrm{i}\epsilon) = P(1/x) \mp \mathrm{i}\pi\delta(x)$. We then obtain

$$h(z) = z - \mathrm{i}\frac{N-2}{2}\omega_0 - \mathrm{i}\frac{2}{3}\frac{d^2\omega_0^2}{\pi} P\int_0^\infty \frac{k^2 g^2(k)\,\mathrm{d}k}{k-\omega_0} + \frac{2}{3}d^2\omega_0^3$$
$$- \mathrm{i}3(N-1)\frac{2}{3}\frac{d^2\omega_0^2}{\pi} P\int_0^\infty \frac{k^2 g^2(k)\,\mathrm{d}k}{k+\omega_0},$$

$$g(z, R) = -\mathrm{i}\frac{d^2}{R^3}\exp(-\mathrm{i}\omega_0 R)(1 + \mathrm{i}\omega_0 R - \omega_0^2 R^2),$$

$$f(z, R) = -\mathrm{i}\frac{d^2}{R^3}\exp(-\mathrm{i}\omega_0 R)(3 + 3\mathrm{i}\omega_0 R - \omega_0^2 R^2).$$

To find the time evolution we have to perform an inverse Laplace transform, so again we are interested in zeros of the determinant of our system of equations for $\bar{b}_i(z)$. If we identify the decay constant for a single atom $\frac{2}{3}d^2\omega_0^3$ with that for an oscillator $e^2\omega_0^2/3m$, it is clearly seen that $g(z, R)$ and $f(z, R)$ in this approximation are exactly equal to those obtained in the oscillator problem with $z = \mathrm{i}\omega_0$: $(1/2\mathrm{i}\omega_0)G(\mathrm{i}\omega_0, R)$, $(1/2\mathrm{i}\omega_0)F(\mathrm{i}\omega_0, R)$. The function $h(z)$ gives the energy shift of the state $|i, \alpha\rangle$ and also its lifetime (which is of course equal to the lifetime of a single excited atom). The total energy shift of this state consists of $N-1$ shifts Δ^- of the energy of the ground state of a single atom and one shift of the energy of the upper state Δ^+.

Δ^- is three times bigger than the shift for two-level atoms obtained for instance by Ackerhalt and Eberly (1974), because there are three (and not one) intermediate states in the proper self-energy diagrams in our case.

Note, however, that Δ^+ does coincide with the conventional expression because our model has a non-degenerate ground state. If we put

$$z_0 = \mathrm{i}(N-2)\omega_0/2 + \mathrm{i}(N-1)\Delta^- + \mathrm{i}\Delta^+ - 1/\tau_0,$$

then our equations can be written in the form

$$(z - z_0)\bar{b}_i(z) + \frac{1}{2\mathrm{i}\omega_0}\sum_{j\neq i}[(G(\mathrm{i}\omega_0, R) - F(\mathrm{i}\omega_0, R)\bar{n}_{ij} \otimes \bar{n}_{ij}) \cdot \bar{b}_j(z)] = \bar{K}_i(z).$$

This is exactly the same system of equations as we obtain in the oscillator problem if we neglect the z-dependence of $G(z, R)$ and $F(z, R)$, i.e. if we seek zeros of the determinant of this system restricted to terms of up to second order in the coupling constant. So all the results that we obtain for oscillators are valid for atoms, if we consider only low excitations over a ground state of the system.

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References

Ackerhalt J R and Eberly J H 1974 *Phys. Rev.* D **10** 3350
Bonifacio R and Banfi P 1975 *Phys. Rev. Lett.* **33** 1259
Dicke R H 1954 *Phys. Rev.* **93** 99
Friedberg R and Hartman S R 1974a *Opt. Commun.* **10** 298
—— 1974b *Phys. Rev.* A **10** 1728
Hamermesh M *Group Theory and its Application to Physical Problems* 1962 (Reading MA: Addison-Wesley)
Milloni P W and Knight P L 1974 *Phys. Rev.* A **10**, 1096
Norton R E and Watson W K 1959 *Phys. Rev.* **116** 1597
Rzążewski K and Żakowicz W 1976 *J. Phys. A: Math. Gen.* **9** 1159
Stephen M J 1964 *J. Chem. Phys.* **40** 669
Weisskopf V F and Wigner E 1930 *Z. Phys.* **63** 54
Żakowicz W 1978 *Phys. Rev.* A **17** 343